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515 Thackeray Hall
University of Pittsburgh, PA 15260

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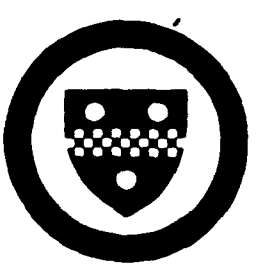

DETECTION OF CHANGE POINTS USING RANK METHODS

B. Q. Miao and L. C. Zhao

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Center for Multivariate Analysis
Fifth Floor Thackeray Hall
University of Pittsburgh
Pittsburgh, PA 15260

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B.Q. Miao and L. C. Zhao

Center for Multivariate Analysis
University of Pittsburgh
Pittsburgh PA 15260

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ABSTRACT

In this paper, the detection and estimation of change points of local parameters are studied by means of localization procedures and rank statistics. These techniques are also applied to detection and estimation of the change points of scale parameters and that of location parameters of directional data.

1. INTRODUCTION

Change point problem arises in many fields and attracts the attention of many authors. The techniques employed to detect and estimate the change point can generally be classified into two categories : parametric (Krishnaiah and Miao .1988), and

nonparametric (Csörgö and Horváth,1988). Bayesian methods also plays a major role, (Zacks, 1983).

In this paper, we concentrate our attention on the problem of detection of change points of location parameter by localization and rank statistics when data are large. Our method is different from, and has some advantages over, the existing methods, such as CUSUM (cumulative sum) and Csörgö and Horváth's non-sequential nonparametric AMOC (at most one change) procedures. First, localized procedures reduce computation. Second, these detecting and estimating procedures require no moment condition, instead we only assume that observed data come from a continuous distribution with a unique median.

2. MODEL AND DETECTING AND ESTIMATING PROCEDURE

Let x(t) be an independent process on the interval (0,1] whose marginal distributions differ only by location parameters. Specifically speaking, there exist $t_0, t_1, \ldots, t_{q+1}$ and $a_1, \ldots, a_{q+1}$ such that $0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = 1$ and

$$x(t) \sim F(x-a_j), \quad \text{if} \quad t_{j-1} < t \leqq t_j, \; j = 1,2,\ldots,q+1, \tag{2.1}$$

with

$$a_j \neq a_{j+1} \quad \text{for} \quad j = 1,\ldots q. \tag{2.2}$$

As usual, $t_1,\ldots,t_q$ are called change points. In practical applications, the number q and the locations of change points are unknown. To estimate q and $t_1,\ldots t_q$, we sample this process sequentially at equal distances, and get x(1/n), x(2/n),... x(n/n). By assumption, x(1/n),...,x(n/n) are independent. Define $k_0^\circ = 0$, $k_{q+1}^\circ = n$, and $k_1^\circ, \ldots, k_q^\circ$ as follows:

$$| k_i^\circ(n)/n - t_i | < 1/n, \quad i = 1,\ldots,q. \tag{2.3}$$

Then we have

$$x(i/n) \sim F(x-a_j), \quad \text{if} \quad k_{j-1}^\circ(n) < i \leqq k_j^\circ(n), \quad j = 1,2,\ldots,q+1, \tag{2.4}$$

From this fact, to estimate $(t_1,\ldots,t_q)$ is equivalent to estimating $(k_1^\circ(n)/n,\ldots, k_q^\circ(n)/n)$. For simplicity, set $x(i/n) = x_i$ and $k_i^\circ(n) = k_i^\circ$. Hereafter, $\alpha_n >> \beta_n$ means $\alpha_n/\beta_n \to \infty$. Let $m = m_n$ and C_n be positive integers such that $n >> m_n >> C_n >> \log n$.

Set $A_{k,m} = \{x_{k-m+1}, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}\}$, $k = m,\ldots,n-m$, k_j be the rank of x_j in $A_{k,m}$, Define

$$y_{k,j} = \begin{cases} 1 \text{ if } x_{k-m+j} \leq m, \\ -1 \text{ otherwise} \end{cases} \quad j = 1,\ldots,m. \tag{2.5}$$

$$S_{k,m} = \sum_{j=1}^{m} y_{k,j}. \tag{2.6}$$

$$D_n = \{k: k = m,\ldots,n-m : S_{k,m}^2/m > C_n\}, \tag{2.7}$$

$$k_{1,n} = \min\{k : k \in D_n\},$$

$$D_{1,n} = \{k : k \in D_n, \ k - k_{1,n} < 3m\}.$$

Next, put

$$k_{2,n} = \min\{k : k \in D_n - D_{1,n}\}$$

$$D_{2,n} = \{k : k \in D_n - D_{1,n}, k-k_{2,n} < 3m\}.$$

Continuing this process, we can define $D_{2,n}, D_{2,n}$ ..., which are easily seen to be nonempty. We have

$$D_n = D_{1,n} + D_{2,n} + \ldots + D_{\hat{q},n}.$$

Define

$$\hat{t}_j = 2^{-1}[k_{j,n} + \max(k : k \in D_{j,n})], \quad j = 1,2,\ldots,\hat{q}.$$

We have the following theorem.

THEOREM. If the distribution F is continuous and has unique median, then $(\hat{q}, \hat{t}_1, \ldots, \hat{t}_{\hat{q}})$ is a strongly consistent estimate of $(q, t_1, \ldots, t_q)$.

3. LEMMAS

To prove the above theorem , we need some lemmas.

LEMMA 1. (Hoeffding, 1963) Let the population C consists of N values $C_1,...,C_N$. Let $x_1,...,x_n$ denote a random sample without replacement from C and let $y_1,...,y_n$ denote a random sample with replacement from C. If the function f(x) is continuous and convex then

$$Ef(\sum_{i=1}^{n} x_i) \leqq Ef(\sum_{i=1}^{n} y_i).$$

LEMMA 2. Let the notation be defined as in section 2 and define

$$B_{k,m} = \{x_{k-m+1},...,x_k\}, \quad k = m,m+1,...,n. \tag{3.1}$$

$$S_{k,m} = \sum_{j=1}^{m} y_{k-m+j}, \tag{3.2}$$

$$B_n = \{k : \exists\, j,\ 1 \leqq j \leqq q+1, \text{ such that } k_{j-1}^{o} + 1 \leqq k - m + 1 < k \leqq k_j^{o}\}. \tag{3.3}$$

Then, we have

$$S_{k,m}^2/m = 0(\log n) \quad \text{a.s.}$$

uniformly for all $k \in B_n$.

Proof. Let $z_{k,1},...,z_{k,m}$ be a random sample with replacement from population $\{1,...,1,-1,...,-1\}$, where the number of 1's and -1's are both m. By lemma 1 for any $t \in (0,1/4)$ and $A > 0$, we have

$$P\{S_{k,m} \geqq A\sqrt{m\log n}\} \leqq \exp\{-tA\sqrt{m\log n}\}Ee^{t\sigma_{k,m}}$$

$$\leqq \exp\{-tA\sqrt{m\log n}\}E\exp\{t\sum_{j=1}^{m} z_{k,j}\}$$

$$= \exp\{-tA\sqrt{m\log n}\}\ (E \exp\{tz_{k_j}\})^m$$

$$= \exp\{-tA\sqrt{m\log n}\}((e^t + e^{-t})/2)^m$$

$$\leqq \exp\{-tA\sqrt{m\log n} + mt^2\, e^t/2\}.$$

Since $m >> \log n$, it is possible for n large to take $t = A\sqrt{(\log n)/m} < 1/4$. It follows

$$P(S_{k,m} \geqq A\sqrt{m\log n}) < \exp\{-0.3A^2\log n\}. \tag{3.4}$$

A similar argument gives

$$P(\sigma_{k,m} \leqq - A\sqrt{m\log n}) < \exp\{-0.3A^2\log n\}. \tag{3.5}$$

The inequalities (3.3) and (3.4) imply

$$\sum_{n=1}^{\infty} P(\sup_{k\in B_n} |S_{k,m}| \geqq A\sqrt{m\log n}) \leqq \sum_{n=1}^{\infty} 2n \exp\{0.3A^2\log n\}.$$

This is a convergence series if $0.3A^2 > 2$, Take $A = 3$, by Borel-Cantelli lemma, with probability one for large n, we have uniformly for all $k \in B_n$,

$$|S_{k,m}| \leqq 3\sqrt{m\log n},$$

$$S^2_{k,m}/m \leqq 9\log n \qquad \text{a.s.}$$

uniformly for all $k \in B_n$.

LEMMA 3. Let $x^*_{i,k}$ denote the i-th sample order statistic in $B_{k,m}$. Let $[\alpha m]$ denote the integer part of αm. Assume that the α-quantile of the continuous distribution F is unique, and is denoted by μ_α. Then

$$x^*_{[\alpha m],k} \to \mu_\alpha + a_j, \quad \text{a.s.}$$

uniformly for all $k \in B_n$, where $k^\circ_{j-1} + 1 \leqq k - m + 1 < k \leqq k^\circ_j$.

Proof. Without loss of generality, we assume that $0 < \alpha < 1$, $k^{o}_{j-1} + 1 \leqq k - m + 1 < k \leqq k^{o}_{j}$ for some fixed j and $a_j = 0$. Write $r = [\alpha m]$, then for any $\varepsilon > 0$,

$$P(|x^{*}_{[\alpha m],k} - \mu_\alpha| \geqq \varepsilon) = P(x^{*}_{r,k} \leqq \mu_\alpha - \varepsilon) + P(x^{*}_{r,k} \geqq \mu_\alpha + \varepsilon)$$

$$\triangleq I_1 + I_2.$$

Set $F(\mu_\alpha - \epsilon) = \alpha - \delta$. By the uniqueness of the α - quantile of F, $\delta > 0$. Without loss of generality, we can assume that $\alpha - \delta > 0$. Since F(x) is continuous, one gets,

$$I_1 = P(x^{*}_{r,k} \leqq \mu_\alpha - \varepsilon)$$

$$= \frac{m!}{(r-1)!(m-r)!} \int_0^{F(\mu_\alpha - \varepsilon)} t^{r-1}(1-t)^{m-r} dt$$

$$= \frac{r.m!}{r!(m-r)!} \int_0^{\alpha - \delta} t^{r-1}(1-t)^{m-r} dt$$

For a fixed $\theta \in (0,1)$, put

$$g_\theta(t) = t^{\theta}(1-t)^{1-\theta}.$$

It is easy to see that g(t) is increasing in t on the interval $[0, \theta]$, which implies that $t^{r-1}(1-t)^{m-r}$ is increasing for $t \in [0, (r-1)/(m-1)]$. Since $r = [\alpha m]$, we get $\alpha - \delta < (r-1)/(m-1)$ for large m. Using Stirling's formula, we obtain

$$I_1 \leqq \frac{rm!}{r!(m-r)!}(\alpha - \delta)^{r-1}(1 - \alpha + \delta)^{m-r}$$

$$\leqq \frac{C_1 m\sqrt{2\pi m}}{\sqrt{2\pi}.2\pi.r(m-r)} \frac{(\alpha-\delta)^{r-1}(1-\alpha+\delta)^{m-r}}{(r/m)^{r}((m-r)/m)^{m-r}}$$

$$\leqq C\sqrt{m}\, q_1^{m} \quad , \tag{3.6}$$

where C_1 and C are constants depending on α, δ, and

$$q_1 = \{(\alpha-\delta)/(\alpha-\delta/2)\}^{\alpha}\{(1-\alpha+\delta)/(1-\alpha+\delta/2)\}^{1-\alpha},$$

$$= g_{\alpha}(\alpha-\delta)/g_{\alpha}(\alpha-\delta/2) < 1.$$

Similarly, one can show that

$$P(x^{*}_{r,k} \geqq \mu + \varepsilon) \leqq C\sqrt{m}\, q_2^m, \tag{3.7}$$

where $0 < q_2 < 1$. Take $q = \max(q_1,q_2)$, then (3.6), (3.7) and $m_n >> \log n$ togather imply

$$\sum_{n=1}^{\infty} P(\sup_{k\in B_n} | x^{*}_{[\alpha m],k} - \mu_{\alpha} - a_j | \geqq \varepsilon) \leqq \sum_{n=1}^{\infty} 2cn\sqrt{m}\, q^m < \infty.$$

By Borel-Cantelli lemma, it follows that

$$x^{*}_{[\alpha m],k} \to \mu_{\alpha} + a_j, \quad \text{a.s.}$$

uniformly for all $k \in B_n$, where $k^{\circ}_{j-1} + 1 \leqq k - m + 1 < k \leqq k^{\circ}_j$.

LEMMA 4. Let $x_i \in A_{k^{\circ}_j,m}$, $j = 1,2,...,q$, $S_{k^{\circ}_j,m}$ be as defined by (2.6). Then there exists a positive λ such that with probability one for n large,

$$S^2_{k^{\circ}_j,m}/m \geqq m\lambda^2, \quad j = 1,...,q.$$

Proof. Let $x_i \in A_{k^{\circ}_j,m}$ for some fixed j. Without loss of generality, we assume $a_{j+1} > a_j$. Take $\lambda \in (0,1/2)$ satisfying the following conditions:

1° the $(1/2-\lambda)$ and $(1/2+\lambda)$ - quantiles of the continuous distribution F(x) are unique.

$$2^{\circ} \quad \mu_{1/2} - \mu_{1/2-\lambda} < (a_{j+1} - a_j)/2, \tag{3.8}$$

$$\mu_{1/2+\lambda} - \mu_{1/2} < (a_{j+1}-a_j)/2. \tag{3.9}$$

Note that λ exists since the median of F is unique and the set {p:0 < p < 1, the p-quantile is not unique } is countable.

Let x^*_{i,k_j° and y^*_{i,k_j° denote the i-th order statistics in $B_{k_j^\circ,\,m}$ and $B_{k_j^\circ+m,m}$ respectively. By lemma 3, for all k_j°, $1 \leqq j \leqq q$,

$$x^*_{[m/2+m],k_j^\circ} \to \mu_{(1/2)+\lambda} + a_j, \quad \text{a.s.} \tag{3.10}$$

$$y^*_{[m/2-m],k_j^\circ} \to \mu_{(1/2)-\lambda} + a_{j+1}, \quad \text{a.s.} \tag{3.11}$$

By (3.8) and (3.9), with probability one for n large,

$$x^*_{[m/2+m],k_j^\circ} < y^*_{[m/2-m],k_j^\circ}, \quad \text{a.s.} \tag{3.12}$$

Now consider the order statistics in the combined sample $B_{k_j^\circ,m} \cup B_{k_j^\circ+m,m} = A_{k_j^\circ,m}$. By (3.12), and the fact that $[m/2 + \lambda m] + [m/2 - \lambda m] \leqq m$, at least $[m/2 + \lambda m]$ x_i's can be found in $B_{k_j^\circ,m}$ which are the first m order statistics of $A_{k_j^\circ,m}$. Therefore, with probability one for n large, we have

$$S_{k_j^\circ,m} \geqq [m/2 + \lambda m] - [m/2 - \lambda m] \geqq \lambda m \quad . \tag{3.13}$$

The lemma is proved.

4. PROOF OF THEOREM AND SOME APPLICATIONS.

Proof of Theorem.

Let k_j^o be defined by (2.3). For fixed j, by lemma 4, we have, with probability one for n large,

$$S^2_{k_j^o,m}/m \geqq m\lambda^2.$$

By the definition of D_n, $k_j^o \in D_n$, j = 1,...,q. Further, by the definition of $D_{j,n}$'s and n >> m, it follows that with probability one for large n, $k_1^o,\ldots,k_q^o$ belong to different $D_{j,n}$'s, which in turn implies

$$\hat{q} \geqq q \quad . \tag{4.1}$$

with probability one for large n. On the other hand, from Lemma 2 and C_n>>log n, it follows that with probability one for n large $S^2_{k,m}/m < C_n$. Further take $0 < \varepsilon < 1/2$ and write

$$K_n = \{k : m \leqq k \leqq n - m, |k/n - t_j| \geqq (m/n)(1+\varepsilon) \quad \text{for } j = 1,\ldots,q\}. \tag{4.2}$$

Then, since n >> m_n, and $A_{k,m}$ consists of iid. random variables, it follows that with probability one for n large

$$D_n \cap K_n = \phi. \tag{4.3}$$

By the definition of $D_{j,n}$ and the fact that $2m(1+\varepsilon) < 3m$, we have with probability one for large n

$$\hat{q} \leqq q. \tag{4.4}$$

Combining (4.1) to (4.4), it follows that $\hat{q} = q$ with probability one for n large, and

$$\lim_{n\to\infty} \hat{t}_j = t_j \qquad j = 1,2,\ldots,q.$$

Thus, the theorem is established.

Our procedure can also be applied to the detection of change points of scale parameters. What we need to do is merely to consider $\log |x(t)|$ instead of $x(t)$.

Likewisely this method can be applied to the detection of change points of location parameters in a circle, i.e. directional data. Lombard (1986) proposed this problem and discussed the related testing and estimating problem when the number of change points is known. Our technique can handle the case in which this number is unknown.

For directional data, fix an origin and an axis and let $\{x(t), 0 \leqq t \leqq 1\}$ be the angle process, where $0 \leqq x(t) < 2\pi$. Assume that $0 = t_0 < t_1 < \ldots < t_q < t_{q+1} = 1$, $t_1,\ldots,t_q$ are the change points, so that $x(t) \sim F(x-a_j)$ if $t_{j-1} < t < t_j$, $j = 1,\ldots, q+1$, where $a_j \neq a_{j+1} \pmod{2\pi}$, $j = 1,\ldots,q$, $a_1,\ldots a_{q+1}$ denote the angular location parameters. The speciality of the angle process is that the angle is always in $[0,2\pi)$. So the angle of $\alpha + \beta$ equals $\alpha + \beta - 2\pi$ if $0 \leqq \alpha,\beta < 2\pi$ and $\alpha + \beta \geqq 2\pi$. A closer look at our proof reveals that the speciality has an effect only on (3.10) and (3.11) when $\mu_{(1/2+\lambda)} + a_j = 0 \pmod{2\pi}$ or $\mu_{(1/2-\lambda)} + a_{j+1} = 0 \pmod{2\pi}$ for some j, $1 \leqq j \leqq q$. Therefore, if continuous distribution $F(x)$ has a unique median and $\mu_{1/2} + a_j \neq 0 \pmod{2\pi}$ for all j, $1 \leqq j \leqq q$, our procedure can be used to detect and estimate the change points of directional data. In practical applications, the origin and axis are chosen after gathering the data, in a way to make the analysis more convenient.

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